Casimir interaction between metal-dielectric metamaterial slabs: Attraction at all macroscopic distances

Mário G. Silveirinha*
Department of Electrical Engineering-Instituto de Telecomunicações, University of Coimbra, 3030 Coimbra, Portugal
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It is theoretically shown that the Casimir force between two planar metamaterial slabs formed by metal-dielectric inclusions with a completely arbitrary shape, size, and material parameters, is invariably attractive at distances a few times larger than the characteristic period of the metamaterials.

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I. INTRODUCTION

In 1948, H. Casimir predicted the attraction between two electrically neutral metallic mirrors due to the quantum fluctuations of the electromagnetic field.[1] The landmark result of Casimir was later extended by Lifshitz to describe the interaction between matter, considering that it can be treated as a continuum characterized by a dielectric function.[2]

The sign and strength of the Casimir force depend on the geometry and properties of the involved materials.[3] The force between two planar uniform dielectric slabs standing in a vacuum is always attractive. However, several physical mechanisms may permit a repulsive force, and in particular recently it was experimentally verified that the interaction between two dielectric objects may be repulsive when they are immersed in a liquid with a suitable dielectric constant.[4] It has also been known for a long time that dielectric and permeable (magnetic) bodies in a vacuum may repel each other.[5] Based on this, several works have studied the possibility of using metal-dielectric metamaterials with exotic optical properties to obtain a repulsive Casimir force, and possibly some form of nanolevitation.[6–9] Such ideas require nanostructuring a bulk dielectric or metal in order to tailor its electromagnetic response. Particularly, it has been suggested that the emergence of artificial magnetism or strong magnetoelectric coupling may provide a route for Casimir repulsive forces.[6,7]

In this paper, we theoretically demonstrate that the Casimir force between two arbitrary planar nanostructured metal-dielectric slabs is invariably attractive at any distance a few times larger than the characteristic period of the arrays. Moreover, we argue that if the structured slabs may be described by a continuous material model (metamaterials) the force is always attractive. Our proof is largely inspired in the beautiful theorem of Ref. 10, which establishes that the Casimir force between two arbitrary dielectric objects with mirror symmetry is necessarily attractive.

II. CASIMIR INTERACTION ENERGY OF TWO PERIODIC DIELECTRIC ARRAYS

In this section, we propose an extension of the scattering (*TGTG*) formalism for the Casimir force introduced in Ref. 10 to the case of two bodies invariant to translations along two given directions of space (periodic dielectric arrays). The developed theory will be used to demonstrate that when the bodies are separated by a macroscopic distance, larger than the characteristic features of the arrays, the Casimir force is always attractive.

We consider two completely arbitrary structured materials (Fig. 1) formed by the repetition of the unit cell $\Omega = \Omega_T \times [-\infty, +\infty]$ where $\Omega_T = \{(x,y) = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 : |\alpha_i| \leq 1/2\}$ is the transverse unit cell and $\mathbf{a}_1$ and $\mathbf{a}_2$ are the primitive vectors, which define the periodicity of both slabs in the *xoy* plane. The region of space $0 < z < d$ is a vacuum. The shape, size, and material parameters of the inclusions in the regions $z < 0$ and $z > d$ are completely arbitrary and are described by a dielectric function $\varepsilon = \varepsilon(\mathbf{r}, \omega)$. The intersection of Ω with the semi-space $z < 0$ ($z > d$) is denoted by A (B). We do not assume any form of translational invariance with respect to z in these regions, and so the thickness of each slab can be finite.

The zero-point energy of the system can be written in terms of the resonant frequencies, ω_n, of the system as: $\mathcal{E} = \Sigma_n \frac{1}{2}\hbar\omega_n$. Since we assume that the structure under analysis is transverse periodic, it is clear that the eigenmodes can be taken as Bloch-Floquet waves, associated with a transverse wave vector $\mathbf{k}_\parallel = (k_x, k_y, 0)$. Hence, we have that

$$\mathcal{E} = \sum_{\mathbf{k}_\parallel} \sum_n \frac{1}{2}\hbar\omega_{n,\mathbf{k}_\parallel}, \tag{1}$$

where $\omega_{n,\mathbf{k}_\parallel}$ are the resonant frequencies associated with the Bloch eigenmodes with transverse wave vector $\mathbf{k}_\parallel$. As shown next, $\omega_{n,\mathbf{k}_\parallel}$ are the zeros of some function $D(\omega, \mathbf{k}_\parallel)$.

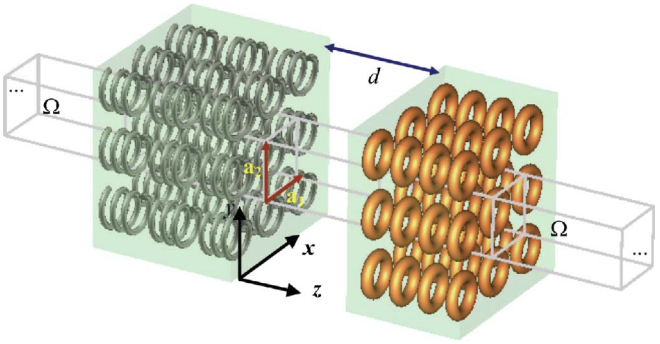

FIG. 1. (Color online) Two planar (transverse periodic) metamaterial slabs formed by metal-dielectric inclusions stand in a vacuum and are separated by a distance d. The framed region represents the basic cell Ω.

In fact, consider a generic Bloch-Floquet electromagnetic mode characterized by the electric field $\mathbf{E}$ and associated with a certain wave vector $\mathbf{k}_\parallel$ and frequency ω. Thus, it satisfies the homogeneous equation:

$$\nabla \times \nabla \times \mathbf{E} - \left(\frac{\omega}{c}\right)^2 \mathbf{E} = \left(\frac{\omega}{c}\right)^2 \chi \mathbf{E}, \tag{2}$$

where $\chi=\varepsilon-1$ is the normalized electric susceptibility of the inclusions, and c is the speed of light in vacuum. Let us introduce the operator $\hat{G}_p=\hat{G}_p(\omega,\mathbf{k}_\parallel)=(\nabla\times\nabla\times-\frac{\omega^2}{c^2})^{-1}$ on $H_{\Omega,\mathbf{k}_\parallel}\to H_{\Omega,\mathbf{k}_\parallel}$, being $H_{\Omega,\mathbf{k}_\parallel}$ the space of (square integrable) vector fields defined over the unit cell that satisfy Bloch-Floquet boundary conditions determined by $\mathbf{k}_\parallel$ in the transverse (x and y) coordinates. Then, Eq. (2) implies that a generic Bloch electromagnetic mode satisfies the Lippmann-Schwinger integral equation, $[\hat{I}-\frac{\omega^2}{c^2}\hat{G}_p(\omega,\mathbf{k}_\parallel)\hat{\chi}(\omega)]\mathbf{E}=0$. Here, $\hat{I}$ is the identity operator and $\hat{\chi}=\hat{\chi}(\omega)$ represents a multiplication operator such that:

$$\hat{\chi}\mathbf{E} = \chi(\mathbf{r};\omega)\mathbf{E}(\mathbf{r}) \tag{3}$$

The electric field in the Lippmann-Schwinger equation is defined over the whole unit cell Ω. However, it is possible to obtain a modified integral equation whose unknown is the restriction $\mathbf{E}_\alpha$ of the electric field to the region $\alpha=A,B$. Indeed, for an observation point $\mathbf{r}$ in A it is clear that, $(\hat{I}_A-\frac{\omega^2}{c^2}\hat{G}_{AA}\hat{\chi}_A)\mathbf{E}_A=\mathbf{E}_A^{inc}$, where $\mathbf{E}_A^{inc}$ is the field that illuminates region A, i.e., the field radiated by the inclusions in region B: $\mathbf{E}_A^{inc}=\frac{\omega^2}{c^2}\hat{G}_{AB}\hat{\chi}_B\mathbf{E}_B$. The subscripts of the operators indicate their domain and range. For example, $\hat{G}_{\alpha\beta}$ is the restriction on $H_{\beta,\mathbf{k}_\parallel}\to H_{\alpha,\mathbf{k}_\parallel}$ of the operator $\hat{G}_p$, being $H_{\alpha,\mathbf{k}_\parallel}$ with $\alpha=A,B$ defined in a similar way as $H_{\Omega,\mathbf{k}_\parallel}$. Likewise, $\hat{\chi}_\alpha$ represents the restriction on $H_{\alpha,\mathbf{k}_\parallel}\to H_{\alpha,\mathbf{k}_\parallel}$ of the multiplication operator $\hat{\chi}$.

Clearly, since the roles of A and B can be interchanged, it is possible to write two analogous equations for an observation point in region B. Using these equations to eliminate $\mathbf{E}_B$, it can be easily shown that $\mathbf{E}_A$ satisfies the following homogeneous integral equation

$$(\hat{I}_A - \hat{T}_A\hat{G}_{AB}\hat{T}_B\hat{G}_{BA})\hat{\chi}_A\mathbf{E}_A = 0, \tag{4}$$

where $\hat{T}_\alpha$ is a mapping on $H_{\alpha,\mathbf{k}_\parallel}\to H_{\alpha,\mathbf{k}_\parallel}$ defined by

$$\hat{T}_\alpha = -\frac{\omega^2}{c^2}\hat{\chi}_\alpha\left(\hat{I}_\alpha - \frac{\omega^2}{c^2}\hat{G}_{\alpha\alpha}\hat{\chi}_\alpha\right)^{-1}. \tag{5}$$

Hence, we see that the dispersion characteristic of the electromagnetic modes associated with the transverse wave vector $\mathbf{k}_\parallel$ may be formally written as $D(\omega,\mathbf{k}_\parallel)=0$, with

$$D(\omega,\mathbf{k}_\parallel) = \det(\hat{I}_A - \hat{T}_A\hat{G}_{AB}\hat{T}_B\hat{G}_{BA}) \tag{6}$$

where "det" stands for the determinant of an operator.[10] In particular, the resonant frequencies $\omega_{n,\mathbf{k}_\parallel}$ that determine the zero point energy of the system are the zeros of $D(\omega,\mathbf{k}_\parallel)$ as we wanted to show.

It is well known that the zero point energy of the system is divergent.[11] However, the interaction energy, which describes the variations of $\mathcal{E}$ with the distance d between the two bodies, is finite and can be conveniently evaluated using the argument principle, similar to the analysis of Refs. 12 and 13. Specifically, it can be easily shown that the contribution of the quantum oscillators associated with a given $\mathbf{k}_\parallel$ to the interaction energy can be written in terms of an integral over the axis of imaginary frequencies ($\omega=i\xi$) as follows:

$$\delta\mathcal{E}_{\mathbf{k}_\parallel} = \frac{\hbar}{2\pi}\int_0^{+\infty} \ln D(i\xi,\mathbf{k}_\parallel)d\xi. \tag{7}$$

The total interaction energy is obviously given by $\delta\mathcal{E}=\Sigma_{\mathbf{k}_\parallel}\delta\mathcal{E}_{\mathbf{k}_\parallel}$. We note that because of the periodicity of the system the transverse wave vector may be assumed to be in the first Brillouin zone, $BZ=\{\beta_1\mathbf{b}_1+\beta_2\mathbf{b}_2:|\beta_i|\le 1/2\}$. The allowed values for $\mathbf{k}_\parallel$ depend on the transverse area of the slabs A_s (following the usual practice, it is assumed that the region of interest is terminated with periodic boundary conditions). Hence, a straightforward analysis (see also the Appendix), shows that the zero temperature Casimir interaction energy per unit of area, $\delta\mathcal{E}/A_s$, is given by:

$$\frac{\delta\mathcal{E}}{A_s} = \frac{\hbar}{(2\pi)^3}\int_{B.Z.} d^2\mathbf{k}_\parallel \int_0^{+\infty} d\xi \ln D(i\xi,\mathbf{k}_\parallel). \tag{8}$$

We note that the Casimir interaction energy is written in terms of the function $D(\omega,\mathbf{k}_\parallel)$ evaluated for imaginary frequencies, whereas the zero-point energy of the system $\mathcal{E}$ is determined by the real valued zeros of the same function.

As discussed, next the above formula corresponds to a generalization of the scattering ($TGTG$) formula for the Casimir interaction introduced in Ref. 10 to the case of a structured material periodic in the transverse (xoy) plane. Indeed, suppose that we take the unit cell Ω as the whole space. It can be easily checked that in such case all the arguments used in the above derivation still hold, except that the summation over $\mathbf{k}_\parallel$ must be suppressed, because when Ω is coincident with the whole space it does not make sense to talk about Bloch modes. Thus, in such case the Casimir interaction energy is written as in Eq. (7):

$$\delta\mathcal{E} = \frac{\hbar}{2\pi}\int_0^{+\infty} \ln D_0(i\xi)d\xi \tag{9}$$

where $D_0(\omega)=\det(\hat{I}_A-\hat{T}_{0A}\hat{G}_{0AB}\hat{T}_{0B}\hat{G}_{0BA})$. The operator $\hat{G}_0$ has formally the same expression as $\hat{G}_p$, i.e., $\hat{G}_0=\hat{G}_0(\omega)=(\nabla\times\nabla\times-\frac{\omega^2}{c^2})^{-1}$, but it is defined over a different domain and satisfies radiation boundary conditions at infinity (while $\hat{G}_p$ satisfies Bloch-periodic boundary conditions in the transverse directions). The operator $\hat{T}_0$ is defined as in Eq. (5), except that $\hat{G}_{\alpha\beta}$ must be replaced by $\hat{G}_{0\alpha\beta}$. Equation (9) is exactly the $TGTG$ formula for the Casimir interaction energy derived in Refs. 10 and 14 using a different approach. We note that the theory of Refs. 10 and 14 assumes implicitly that the bodies of interest are uniform (χ_A and χ_B are constants), whereas our formula is valid even for nonuniform bodies. It is important to mention that for nonuniform bodies $\chi=\chi(\mathbf{r};\omega)$ and in that case the operators $\hat{\chi}_\alpha$ do not commute

with the operators $\hat{G}_{\alpha\beta}$. In particular, the order of the operators in the definition of $\hat{T}$ [Eq. (5)] is important.

Our derivation the *TGTG* scattering formula is based on the argument principle and thus, strictly speaking, is only valid in case of very low material losses (because only then the resonance frequencies $\omega_{n,\mathbf{k}_\parallel}$ are real valued). However, formula (8) can be as well derived using the theory of Kenneth and Klich [Eq. (9)], which applies even for lossy structures. This alternative derivation of Eq. (8) is presented in Appendix.

It is interesting to note that the operators $\hat{T}_\alpha$ have a simple physical meaning for ω real valued. In fact, it should be clear from our derivation, that $\hat{T}_\alpha$ relates the electric density of current $\mathbf{J}_\alpha = -i\omega\varepsilon_0\chi\mathbf{E}$ induced in region α with the corresponding incident field $\mathbf{E}_\alpha^{inc}$ through the relation: $\mathbf{J}_\alpha = \frac{1}{-i\omega\mu_0}\hat{T}_\alpha\mathbf{E}_\alpha^{inc}$. Notice that $\hat{T}_\alpha$ depends exclusively on the properties of the body α. The interaction between the two bodies is described by the operators $\hat{G}_{AB}$ and $\hat{G}_{BA}$.

An important property of the operator $\hat{G}_p = \hat{G}_p(\omega, \mathbf{k}_\parallel)$ is that it is periodic in $\mathbf{k}_\parallel$. This follows immediately from the fact $\hat{G}_p$ is defined over the space $H_{\Omega,\mathbf{k}_\parallel}$, which is coincident with $H_{\Omega,\mathbf{k}_\parallel+j_1\mathbf{b}_1+j_2\mathbf{b}_2}$, where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the primitive vectors of the reciprocal lattice. In particular, this shows that the operators $\hat{G}_{\alpha\beta}$, and thus also $D(\omega, \mathbf{k}_\parallel)$, are periodic in $\mathbf{k}_\parallel$. Hence, even though in Eq. (8) the integration is done only over the first Brillouin zone the information about all the other zones is implicitly contained in $D(\omega, \mathbf{k}_\parallel)$.

It is also useful to note that the action of $\hat{G}_p$ over a generic vector field $\mathbf{E}$ can be written in terms of the Bloch-periodic Green dyadic $\bar{\bar{G}}_p = \bar{\bar{G}}_p(\mathbf{r}, \mathbf{r}'; \omega, \mathbf{k}_\parallel)$ as follows:

$$\hat{G}_p\mathbf{E} = \int_\Omega \bar{\bar{G}}_p(\mathbf{r}, \mathbf{r}'; \omega, \mathbf{k}_\parallel) \cdot \mathbf{E}(\mathbf{r}')d^3\mathbf{r}' \tag{10}$$

The Green dyadic $\bar{\bar{G}}_p$ is determined by the radiation field of an array of point sources in free space, and satisfies

$$\nabla\times\nabla\times\bar{\bar{G}}_p - \frac{\omega^2}{c^2}\bar{\bar{G}}_p = \bar{\bar{\mathbf{I}}}\sum_{I=(i_1,i_2)}\delta(\mathbf{r}-\mathbf{r}'-\mathbf{r}_\mathbf{I})e^{i\mathbf{k}_\parallel\cdot(\mathbf{r}-\mathbf{r}')}, \tag{11}$$

where $\mathbf{r}_\mathbf{I} = i_1\mathbf{a}_1 + i_2\mathbf{a}_2$ is a generic lattice point. It is quite evident from the above relations that $\hat{G}_p$ is indeed a periodic function of $\mathbf{k}_\parallel$.

For future reference, we note that the Green dyadic $\bar{\bar{G}}_p$ can be written in terms of a scalar potential Φ_p as follows:

$$\bar{\bar{G}}_p(\mathbf{r}, \mathbf{r}'; i\xi, \mathbf{k}_\parallel) = \left(\bar{\bar{\mathbf{I}}} - \frac{c^2}{\xi^2}\nabla\nabla\right)\Phi_p(\mathbf{r}, \mathbf{r}'), \tag{12}$$

where Φ_p is a Bloch-periodic Green function that has the following spectral representation (see for example, Ref. 15):

$$\Phi_p(\mathbf{r}, \mathbf{r}'; i\xi, \mathbf{k}_\parallel) = \sum_{\mathbf{J}=(j_1,j_2)}\frac{e^{-\gamma_\mathbf{J}|z'-z|}}{2\gamma_\mathbf{J}A_{cell}}e^{i\mathbf{k}_\mathbf{J}\cdot(\mathbf{r}-\mathbf{r}')}, \tag{13}$$

where $A_{cell} = |\mathbf{a}_1\times\mathbf{a}_2|$ is the area of the transverse cell, $\mathbf{k}_\mathbf{J} = \mathbf{k}_\parallel + j_1\mathbf{b}_1 + j_2\mathbf{b}_2$, j_1 and j_2 are generic integers, and $\gamma_\mathbf{J} = \sqrt{\mathbf{k}_\mathbf{J}\cdot\mathbf{k}_\mathbf{J} + \xi^2/c^2}$.

III. ASYMPTOTIC BEHAVIOR OF THE OPERATORS AND CONVERGENCE OF THE INTERACTION ENERGY INTEGRAL

Here, we show that the integral in Eq. (8) is well defined and yields a finite value for the Casimir interaction energy. A somehow related discussion for the case in which the system is not periodic is presented in Ref. 14.

To begin with, we note that in Eqs. (12) and (13) we have $\gamma_\mathbf{J} \sim \xi/c$ and thus, for $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (x', y', z')$ such that $z < 0$ and $z' > d$, the Green dyadic $\bar{\bar{G}}_p$ converges exponentially to zero as $\xi \to \infty$. Using Eq. (10), it should be clear that this implies that the operators $\hat{G}_{AB}$ and $\hat{G}_{BA}$ converge as well exponentially to zero as $\xi \to \infty$.

On the other hand, it is well known that because of the passivity and causality of the materials,[16] $\chi(\mathbf{r}; i\xi) > 0$ in the dielectrics, and thus $\hat{\chi}_\alpha$ are nonnegative operators for imaginary frequencies (i.e., $\langle\mathbf{E}|\hat{\chi}_\alpha|\mathbf{E}\rangle \geq 0$ for a generic vector field $\mathbf{E}$). Hence, for $\omega = i\xi$ we can write $\hat{T}_\alpha$ defined by Eq. (5), as follows:

$$\hat{T}_\alpha = \hat{\chi}_\alpha^{1/2}\hat{S}_\alpha\hat{\chi}_\alpha^{1/2} \tag{14}$$

$$\hat{S}_\alpha = \frac{\xi^2}{c^2}\left(\hat{I}_\alpha + \frac{\xi^2}{c^2}\hat{\chi}_\alpha^{1/2}\hat{G}_{\alpha\alpha}\hat{\chi}_\alpha^{1/2}\right)^{-1}. \tag{15}$$

It can be easily checked using Eq. (10) that $\hat{G}_{\alpha\alpha}$ is self-adjoint (Hermitian) for imaginary frequencies: $\hat{G}_{\alpha\alpha} = \hat{G}_{\alpha\alpha}^\dagger$. Since, $\hat{\chi}_\alpha$ is also evidently self-adjoint for imaginary frequencies it follows that both $\hat{T}_\alpha$ and $\hat{S}_\alpha$ are self-adjoint for $\omega = i\xi$.

We also note that $\hat{G}_p$ is positive definite for $\omega = i\xi$, i.e., $\hat{G}_p > 0$. This follows from the fact that $\hat{G}_p^{-1} = \nabla\times\nabla\times + \frac{\xi^2}{c^2}$ is obviously Hermitian and positive definite. Hence, we see that $\frac{c^2}{\xi^2}\hat{S}_\alpha$ is a bounded positive operator, and in particular $\hat{T}_\alpha$ has the same asymptotic behavior (for large ξ) as $\chi_\alpha\xi^2/c^2$. But since the materials do not have an electric response for very large (imaginary) frequencies,[16] we also know that $\chi_\alpha \to 0$. Hence, remembering that $\hat{G}_{AB}$ and $\hat{G}_{BA}$ converge exponentially to zero, we finally conclude that $\hat{T}_A\hat{G}_{AB}\hat{T}_B\hat{G}_{BA}$ converges as well exponentially to zero as $\xi \to \infty$. This property ensures that the integral in Eq. (8) is well defined and is convergent, as we wanted to show.

IV. $\hat{M}\hat{M}^\dagger$ STRUCTURE

The *TGTG* formalism enables one to write in a formal way the Casimir interaction energy in terms of the geometry of the system and of the material parameters. In order to

study the sign of the Casimir force, it is convenient to make the dependence on the distance between the two bodies, d, explicit in the definition of the operators.

To this end, we may assume for simplicity that the body A is held fixed and study the variation of the interaction energy as the position of the body B varies. Thus, the set B may be regarded as a function of d, i.e., $B=B(d)$. As mentioned above, we want to make the dependence of $\delta\mathcal{E}$ on d explicit, and thus it is desirable that the domains and ranges of all the operators are independent of d. Similar to Ref. 10, this can be achieved by introducing a unitary operator $\hat{\mathfrak{I}}:H_{C,\mathbf{k}_\parallel} \rightarrow H_{B,\mathbf{k}_\parallel}$ such that $\hat{\mathfrak{I}}\mathbf{E}=\mathbf{E}(T_{-\mathbf{d}}.\mathbf{r})$, where by definition $C \equiv B(0)$ and $T_{\mathbf{d}}$ is a mapping (translation) such that $T_{\mathbf{d}}:\mathbf{r} \rightarrow \mathbf{r}+\mathbf{d}$ and $\mathbf{d}=d\hat{\mathbf{z}}$. The region C corresponds to the configuration of the slab in the region $z>0$ when d vanishes. It can now be easily checked that $\hat{G}_{AB}\hat{T}_B\hat{G}_{BA}=\hat{G}_{AC}\hat{T}_C\hat{G}_{CA}$, where $\hat{T}_C=\hat{\mathfrak{I}}^\dagger\hat{T}_B\hat{\mathfrak{I}}=\hat{T}_{B(0)}$ and $\hat{G}_{AC}=\hat{G}_{AB}\hat{\mathfrak{I}}$. The operator $\hat{G}_{AC}(i\xi,\mathbf{k}_\parallel)$ can be written explicitly as

$$\hat{G}_{AC}\mathbf{E}=\int_C \bar{\bar{G}}_p(\mathbf{r},\mathbf{r}'+\mathbf{d};i\xi,\mathbf{k}_\parallel)\,.\,\mathbf{E}(\mathbf{r}')d^3\mathbf{r}'. \tag{16}$$

It can be easily shown that for imaginary frequencies, $\omega=i\xi$, we have that $\hat{G}_{CA}=\hat{G}_{AC}^\dagger$. Hence, for $\omega=i\xi$ we can write D as follows:

$$D(i\xi,\mathbf{k}_\parallel)=\det(\hat{I}_A-\hat{T}_A\hat{G}_{AC}\hat{T}_C\hat{G}_{AC}^\dagger). \tag{17}$$

It is interesting to note that $\hat{T}_A$ and $\hat{T}_C$ are completely independent of the distance between the bodies, and convey all the information about their geometry and material parameters. On the other hand, the operator $\hat{G}_{AC}$, which describes the electromagnetic wave propagation in a vacuum, is totally independent of the geometry and permittivity of the materials, and evidently depends on d, as shown explicitly in Eq. (16).

The determinant of a self-adjoint operator is equal to the product of the respective eigenvalues. However, it can be easily checked that $\hat{I}_A-\hat{T}_A\hat{G}_{AC}\hat{T}_C\hat{G}_{AC}^\dagger$ does not have this property. Despite this difficulty it is possible, as shown next, to write D as the determinant of a self-adjoint operator.

Indeed, if $\hat{Q}$ is a generic operator and $\hat{P}$ is a nonnegative operator $\hat{P}\geq 0$ we have that:[20]

$$\det(\hat{I}_A-\hat{P}\hat{Q})=\det(\hat{I}_A-\hat{P}^{1/2}\hat{Q}\hat{P}^{1/2}). \tag{18}$$

Using now Eq. (14) and the fact that $\hat{\chi}_\alpha\geq 0$ for imaginary frequencies (see Sec. III), it is evident that D can be written in terms of the determinant of an Hermitian operator as follows:

$$D(i\xi,\mathbf{k}_\parallel)=\det(\hat{I}_A-\hat{M}\hat{M}^\dagger). \tag{19}$$

In the above, $\hat{M}=\hat{S}_A^{1/2}\hat{\chi}_A^{1/2}\hat{G}_{AC}\hat{\chi}_C^{1/2}\hat{S}_C^{1/2}$, $\hat{\chi}_C=\hat{\chi}_{B(0)}$, and $\hat{S}_A$ and $\hat{S}_C$ are defined as in Eq. (15). To obtain Eq. (19) we took into account that for $\omega=i\xi$, $\hat{S}_\alpha$ are self-adjoint nonnegative operators, $\hat{S}_\alpha\geq 0$.

The eigenvalues of the Hermitian operator $\hat{M}\hat{M}^\dagger$ are such that $0\leq\lambda_i(i\xi,\mathbf{k}_\parallel)<1$. The lower bound is evident, while the upper bound can be justified by noting that the operator $\hat{G}_{AC}$, which determines the interaction between the two slabs, decays exponentially to zero as $\xi\rightarrow\infty$, and hence, $\hat{M}$ has the same property. Hence, for sufficiently large ξ the eigenvalues of $\hat{M}\hat{M}^\dagger$ are clearly less than unity. On the other hand, the eigenvalues of $\hat{I}_A-\hat{M}\hat{M}^\dagger$ (i.e., $1-\lambda_i$) vary continuously with ξ, and cannot cross zero because this would imply that for some $\xi>0$ one would have $D(i\xi,\mathbf{k}_\parallel)=0$, which is impossible on physical grounds given that the zeros of D are associated with the eigenmodes of the structure, and for passive materials these must necessarily occur for complex frequencies such that $\mathrm{Im}(\omega)\leq 0$. Therefore, the eigenvalues $1-\lambda_i$ must have the same sign for all ξ, and thus, must be positive real numbers.

V. PROOF OF ATTRACTION AT MACROSCOPIC DISTANCES

Using the theoretical formalism developed in the previous sections, next we prove that when the distance between the slabs is "macroscopic," the Casimir force is necessarily attractive. To begin with, we note that the only operator in the definition of $\hat{M}$ that depends on d is $\hat{G}_{AC}$, as shown explicitly in Eq. (16). Using Eqs. (12) and (13) it is obvious that the kernel of the integral operator $\hat{G}_{AC}$ is such that:

$$\bar{\bar{G}}_p(\mathbf{r},\mathbf{r}'+\mathbf{d})=\left(\bar{\bar{\mathbf{I}}}-\frac{c^2}{\xi^2}\nabla\nabla\right)\Phi_p(\mathbf{r},\mathbf{r}'+\mathbf{d})$$

$$\Phi_p(\mathbf{r},\mathbf{r}'+\mathbf{d})=\sum_{\mathbf{J}=(j_1,j_2)}\frac{e^{-\gamma_{\mathbf{J}}(d+z'-z)}}{2\gamma_{\mathbf{J}}A_{cell}}e^{i\mathbf{k}_{\mathbf{J}}\cdot(\mathbf{r}-\mathbf{r}')}. \tag{20}$$

We took into account that $z'-z>0$ when $\mathbf{r}\in A$ and $\mathbf{r}'\in C$, which is the case of interest because $\hat{G}_{AC}$ is an operator on $H_{C,\mathbf{k}_\parallel}\rightarrow H_{A,\mathbf{k}_\parallel}$.

For $\mathbf{k}_\parallel$ in the Brillouin zone the propagation constant $\gamma_{\mathbf{J}}$ increases with $\mathbf{J}=(j_1,j_2)$. Thus, the term that contributes mostly to the Green function Φ_p is the one with $\mathbf{J}=\mathbf{0}$. Moreover, the contribution of the term associated with the index $\mathbf{J}$ may be estimated as smaller than that of $\mathbf{J}=\mathbf{0}$ by a factor of $e^{-|\mathbf{k}_\mathbf{J}|d}\sim e^{-|\mathbf{J}|2\pi d/a}$, where a is the characteristic period (lattice constant) of the transverse lattice. Hence, provided d is a few times larger than a, let's say $d>d_0$, it follows that the contribution of all terms with $\mathbf{J}\neq 0$ is negligible as compared to that of $\mathbf{J}=0$. Therefore, in such circumstances, we may write with negligible error that:

$$\Phi_p(\mathbf{r},\mathbf{r}'+\mathbf{d})\approx e^{-\gamma_0 d}\frac{e^{-\gamma_0(z'-z)}}{2\gamma_0 A_{cell}}e^{i\mathbf{k}_\parallel\cdot(\mathbf{r}-\mathbf{r}')}. \tag{21}$$

In particular, it follows that for $d>d_0$ the operator $\hat{G}_{AC}$ verifies $\hat{G}_{AC}\approx e^{-\gamma_0(d-d_0)}\hat{G}_{AC,0}$, where $\hat{G}_{AC,0}$ is equal to $\hat{G}_{AC}$ calculated for $d=d_0$. Likewise, for $d>d_0$ the operator $\hat{M}$ may also be written as $\hat{M}\approx e^{-\gamma_0(d-d_0)}\hat{M}_0$, where $\hat{M}_0=\hat{M}|_{d=d_0}$. This

shows that the eigenvalues of $\hat{M}\hat{M}^{\dagger}$ satisfy the relation:

$$\lambda_i(i\xi,\mathbf{k}_{\parallel}) \approx e^{-2\gamma_0(d-d_0)}\lambda_{0,i}(i\xi,\mathbf{k}_{\parallel}), \tag{22}$$

where $\lambda_{0,i}=\lambda_{0,i}(i\xi,\mathbf{k}_{\parallel})$ are the eigenvalues of $\hat{M}_0\hat{M}_0^{\dagger}$. Since the determinant of an Hermitian operator is equal to the product of the respective eigenvalues, we find, using Eq. (19), that for $d>d_0$,

$$\ln D(i\xi,\mathbf{k}_{\parallel}) = \sum_i \ln(1-e^{-2\gamma_0(d-d_0)}\lambda_{0,i}). \tag{23}$$

But, as proven in Sec. IV, the eigenvalues of $\hat{M}_0\hat{M}_0^{\dagger}$ satisfy $0\leq\lambda_{0,i}<1$, and thus it is trivial to check that the interaction energy Eq. (8) increases with increasing d, i.e., for $d>d_0$ the Casimir force is attractive, being d_0 some distance comparable with the transverse lattice constant.

The derived results are exact if the structured materials are uniform in the transverse directions, i.e., if $\varepsilon=\varepsilon(z,\omega)$. Indeed, in such conditions we can choose the (transverse) period a as small as we may wish, and in particular, letting $a\to 0$ formula (21) becomes an exact identity. This is consistent with the results of Ref. 17, which showed using an approach related to ours that repulsion between two objects is forbidden if one of them is an infinite slab with translational symmetry.

It is important to mention that our theory does not preclude the possibility of repulsion between the two structured slabs at distances comparable with the lattice constant (i.e., $d<d_0$) when the material parameters depend on the transverse coordinates. We will not attempt to give a mathematically rigorous upper bound for d_0, since it depends in a complicated manner on the geometry of the structured slabs. However, we emphasize that the electromagnetic interaction between the two slabs is completely described by the operator $\hat{G}_{AC}$, and that the approximation of Eq. (21) is very accurate for imaginary frequencies when d is a few times larger than the transverse period, being the estimated relative error $e^{-2\pi d/a}$. In fact, from a physical point of view the approximation of Eq. (21) is equivalent to say that the slabs cannot sense the granularity of each other, so that the wave propagation in the vacuum region is described by the dominant Fourier harmonic with propagation constant $\gamma_0=\sqrt{k_{\parallel}^2+(\xi/c)^2}$. It is interesting to point out that typically the main contribution to the Casimir interaction energy comes from low values of ξ/c and values of $\mathbf{k}_{\parallel}$ close to the center of the Brillouin zone, because it is for those values that $\gamma_0=\sqrt{k_{\parallel}^2+(\xi/c)^2}$ is minimal and thus that the electromagnetic interaction between the slabs is stronger.

Moreover, if the interaction between the periodically structured slabs can be described using effective medium theory it is clear that the bodies cannot sense the granularity of each other in the x and y directions (otherwise effective medium theory would not apply), and hence also in these conditions Eq. (21) is exact. Or in other words, if the considered metal-dielectric slabs behave as continuous media (possibly characterized by some exotic effective parameters in the real frequency axis), the Casimir force is invariably attractive. In fact, we would like to point out that the unusual optical effects of a metamaterial such as optical magnetism

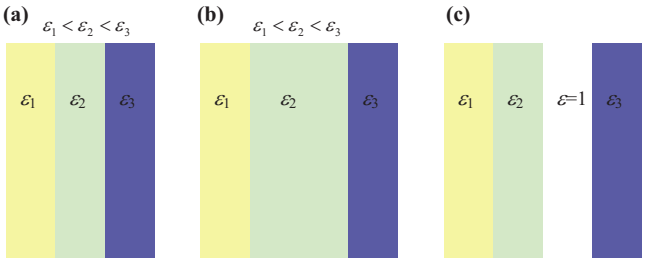

FIG. 2. (Color online) Different configurations of uniform dielectric slabs. The theory of this work forbids a transition (a) $\to$(c), but not a transition (a)$\to$(b).

are a consequence of the structures it consists of ("inclusions"), and that such (near-field) interactions between the inclusions are fully described by the $\hat{T}_{\alpha}$ operators, which are treated without any approximation in our theory. On the other hand, within the scope of any effective medium theory, the interaction between the two slabs is intrinsically a far-field interaction relative to the scale of the lattice constant and is described by $\hat{G}_{AC}$. The fact that such interaction is effectively in the far-field (relative to the scale of the transverse lattice constant) justifies that we can indeed use the approximation implicit in Eq. (21), and implies that the Casimir force is attractive.

It is also important to emphasize that the results of this work apply only to displacements of the material slabs with respect to a vacuum. This should be clear from the fact that we are calculating the variation of the energy of the system when the distance d between the slabs is changed, and in both the initial and final configurations the space in between the slabs is a vacuum (naturally, the system evolves to a configuration that minimizes its energy). However, the theory does not preclude repulsion when the two bodies are embedded in another material (e.g., a fluid), which, as it is well known,[4,18] is physically possible.

In fact, consider the scenarios depicted in Fig. 2, which represent several configurations of uniform dielectric slabs with different permittivities. Our result forbids that regardless of the values of the permittivities ε_1, ε_2, and ε_3 the system may evolve from configuration (a) to configuration (c), because the internal energy associated with configuration (c) is greater than that of configuration (a). However, very importantly, the theory does not forbid that configuration (a) evolves to configuration (b) when $\varepsilon_1<\varepsilon_2<\varepsilon_3$. In fact, such transition corresponds to a displacement with respect to a background material that is not a vacuum. Such displacement may occur if the middle layer is a fluid (i.e., if the two slabs ε_1, and ε_3 are embedded in a fluid). Interestingly, the fluid may be very well a metamaterial, even if the inclusions stand in a vacuum! The key point is that in the transition (a) $\to$(b) additional "inclusions" must be displaced into the middle region to fill the extra space corresponding to the displacement of the slabs.

Finally, we note that our result does not invalidate the mechanisms of repulsion considered in either Ref. 6 or Ref. 9, because the former is based on active materials whereas the latter is based on materials with intrinsic magnetism. However, our theory opposes Ref. 7, which showed that the

interaction between two isotropic chiral metamaterials may be repulsive for a sufficiently large chirality parameter. In fact, at least as long as the material properties are obtained by structuring either metals or dielectrics, the Casimir force is certainly attractive at all macroscopic distances, and thus when effective medium theory applies. In fact, the analysis of Ref. 19 indicates that the large chirality parameter considered in Ref. 7 may be unphysical.

VI. CONCLUSION

In this work, we derived a generalized *TGTG* scattering formula which applies to arbitrary transverse periodic metal-dielectric systems. Using the developed theoretical formalism, it was shown that the Casimir interaction between two planar transverse periodic metal-dielectric structured slabs separated by a vacuum is necessarily attractive at any macroscopic distance.

The proof is based on the transformation of the *TGTG* structure of Ref. 10 into an equivalent $\hat{M}\hat{M}^{\dagger}$ structure. Such transformation is only possible at imaginary frequencies $\omega=i\xi$, and relies on the fact that $\hat{T}_{\alpha}\geq 0$, which is a consequence of $\hat{G}_{\alpha\alpha}$ and $\hat{\chi}_{\alpha}$ having the same property due to the passivity and causality of the materials. The operators $\hat{T}_{\alpha}$ describe the near-field interactions between the inclusions and convey all the information about the effective optical properties of the metamaterials. These operators are considered without any approximations in our theory. The last ingredient of the proof is the fact that in the "far-field" (with respect to the transverse lattice constant) the Green function cannot resolve the structure of the slabs, and hence the approximation Eq. (21) can be used to describe their interaction.

The obtained result is valid at *any* distance provided the material parameters are independent of the transverse x and y coordinates. Moreover, the theory is also valid when the interaction between the slabs can be described using effective medium theory, so that the slabs do not see the granularity of one another. Our theory restricts considerably the possibility of obtaining Casimir repulsion by using metal-dielectric planar structured materials separated by a vacuum.

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APPENDIX: TGTG FORMULA FOR THE INTERACTION BETWEEN TWO PERIODIC DIELECTRIC ARRAYS

Here, we use the theory of Kenneth and Klich[10] to obtain a alternative derivation of Eq. (8). In our notations the *TGTG* formula of Ref. 10 reads (all the operators are evaluated for $\omega=i\xi$):

$$\delta\mathcal{E}=\frac{\hbar}{2\pi}\int_0^{+\infty} d\xi \ln\det(\hat{I}-\hat{T}_{0A}\hat{G}_{0AB}\hat{T}_{0B}\hat{G}_{0BA}), \quad \text{(A1)}$$

where $\hat{T}_{0\alpha}=-\frac{\omega^2}{c^2}\hat{\chi}_{\alpha}(\hat{I}_{\alpha}-\frac{\omega^2}{c^2}\hat{G}_{0\alpha\alpha}\hat{\chi}_{\alpha})^{-1}$ and $\hat{G}_0=(\nabla\times\nabla\times-\frac{\omega^2}{c^2})^{-1}$. Formula (A1) cannot be directly applied to a periodic system because in such case the number of objects is infinite, and thus the interaction energy is also infinite. However, as shown next, it is possible to calculate the interaction energy per unit of area.

To this end, consider the geometry of the main text and define the set $\Omega_N=\{(x,y,z)=\alpha_1\mathbf{a}_1+\alpha_2\mathbf{a}_2+z\hat{\mathbf{z}}:|\alpha_i|\leq N/2,-\infty<z<+\infty\}$ where $\mathbf{a}_1$ and $\mathbf{a}_2$ are the primitive vectors of the transverse lattice and N is a large integer number. It should be clear that Ω_N consists of exactly $N\times N$ primitive cells. In order to calculate the Casimir interaction energy in Ω_N we can assume that Ω_N corresponds to a "cavity" terminated with periodic boundary conditions. Thus, the Casimir energy in Ω_N can be calculated using Eq. (A1) provided the set A (B) is understood as the intersection of the region $z<0$ ($z>d$) with Ω_N, and in addition provided it is understood that the domain of all the operators is H_{Ω_N}, being H_{Ω_N} the space of square integrable vector fields that are periodic in Ω_N. Notice that the restriction of $\hat{G}_0$ to the space H_{Ω_N} corresponds to a Green function that verifies periodic boundary conditions over the boundary of Ω_N.

The key result is that H_{Ω_N} can be written as the direct sum of $N\times N$ subspaces as follows:

$$H_{\Omega_N}=\tilde{H}_{\Omega,\mathbf{k}_{\parallel,1}}\oplus\tilde{H}_{\Omega,\mathbf{k}_{\parallel,2}}\oplus\ \ldots\ \oplus\tilde{H}_{\Omega,\mathbf{k}_{\parallel,N\times N}}, \quad \text{(A2)}$$

where $\mathbf{k}_{\parallel,i}$ is a wave vector of the form $\mathbf{k}_{\parallel}=\frac{j_1}{N}\mathbf{b}_1+\frac{j_2}{N}\mathbf{b}_2$ with $j_1,j_2=0,\ldots N-1$, being $\mathbf{b}_1$ and $\mathbf{b}_2$ the primitive vectors of the reciprocal transverse lattice. The vector space $\tilde{H}_{\Omega,\mathbf{k}_{\parallel,i}}$ is a subspace of H_{Ω_N} such that its elements satisfy the Bloch-Floquet condition (associated with the wave vector $\mathbf{k}_{\parallel,i}$) in the unit cell Ω. The important point is that because the material is periodic, $\tilde{H}_{\Omega,\mathbf{k}_{\parallel,i}}$ are eigensubspaces of the operator $\hat{I}-\hat{T}_{0A}\hat{G}_{0AB}\hat{T}_{0B}\hat{G}_{0BA}$. Thus it follows that its determinant is given by:

$$\det(\hat{I}-\hat{T}_{0A}\hat{G}_{0AB}\hat{T}_{0B}\hat{G}_{0BA})=\prod_i \det([\hat{I}-\hat{T}_{0A}\hat{G}_{0AB}\hat{T}_{0B}\hat{G}_{0BA}]_{\tilde{H}_{\Omega,\mathbf{k}_{\parallel,i}}}), \quad \text{(A3)}$$

where $[\ldots]_{\tilde{H}_{\Omega,\mathbf{k}_{\parallel,i}}}$ represents the restriction of the operator to the proper subspace $\tilde{H}_{\Omega,\mathbf{k}_{\parallel,i}}$. Notice that if $\hat{A}$ is a generic operator $\det(\hat{A})$ can be seen as the determinant of its representation matrix with respect to some basis of the space. The determinant of the restriction of $\hat{A}$ to a proper subspace $\tilde{H}_{\Omega,\mathbf{k}_{\parallel,i}}$ is defined similarly, being the difference that now we must consider a basis $\tilde{H}_{\Omega,\mathbf{k}_{\parallel,i}}$ (instead of H_{Ω_N}). Thus, Eq. (A3) merely states that the representation matrix of the operator can be chosen such that its formed by block matrices distributed along the main diagonal, being each block matrix associated with the restriction of the operator to a proper subspace.

Next, we note that it is possible to map the subspace $\tilde{H}_{\Omega,\mathbf{k}_{\parallel}}$ into the subspace $H_{\Omega,\mathbf{k}_{\parallel}}$ defined in Sec. II through a trivial unitary transformation $\mathfrak{I}:\tilde{H}_{\Omega,\mathbf{k}_{\parallel}}\rightarrow H_{\Omega,\mathbf{k}_{\parallel}}$ such that for a generic vector field $\mathbf{E}$ one has $\mathfrak{I}\mathbf{E}=\mathbf{E}|_{\Omega}$. Notice that the sub-

space $\tilde{H}_{\Omega,\mathbf{k}_\parallel}$ is virtually the same as the subspace $H_{\Omega,\mathbf{k}_\parallel}$, being the difference between them that in the former the vector fields are defined over the set Ω_N whereas in the latter they are defined over Ω. A straightforward analysis shows that $\mathfrak{I}[\hat{I}-\hat{T}_{0A}\hat{G}_{0AB}\hat{T}_{0B}\hat{G}_{0BA}]_{\tilde{H}_{\Omega,\mathbf{k}_\parallel}}\mathfrak{I}^\dagger=\hat{I}-\hat{T}_A\hat{G}_{AB}\hat{T}_B\hat{G}_{BA}$ being $\hat{T}_\alpha=\hat{T}_\alpha(\omega,\mathbf{k}_\parallel)$ and $\hat{G}_{\alpha\beta}=\hat{G}_{\alpha\beta}(\omega,\mathbf{k}_\parallel)$ the operators defined in Sec. II. Therefore from Eq. (A3) we obtain that:

$$\det(\hat{I}-\hat{T}_{0A}\hat{G}_{0AB}\hat{T}_{0B}\hat{G}_{0BA})=\prod_i \det(\hat{I}-\hat{T}_A\hat{G}_{AB}\hat{T}_B\hat{G}_{BA}|_{i\xi,\mathbf{k}_{\parallel,i}}). \tag{A4}$$

Substituting the above result into Eq. (A1) it is found that

$$\delta\mathcal{E}=\frac{\hbar}{2\pi}\sum_{\mathbf{k}_{\parallel,i}}\int_0^{+\infty} d\xi \ln\det([\hat{I}-\hat{T}_A\hat{G}_{AB}\hat{T}_B\hat{G}_{BA}]|_{i\xi,\mathbf{k}_{\parallel,i}}) \tag{A5}$$

Finally, we may let $N\to\infty$ and note that in such case the summation over $\mathbf{k}_{\parallel,i}$ becomes an integral over the first Brillouin zone. Taking into account that the element of area in the Brillouin zone is given by $d^2\mathbf{k}_\parallel=\frac{(2\pi)^2}{A_{cell}N^2}$, where A_{cell} is the area of the transverse unit cell in the space domain, and noting that $A_s=A_{cell}N^2$ is the area of structure's cross-section in Ω_N we readily obtain Eq. (8).

*mario.silveirinha@co.it.pt

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[20] The proof for a positive operator $\hat{P}>0$ is straightforward; the result is also valid for non-negative operators because such operators can be perturbed to become positive, and because the det(.) function is continuous.